MATH 2060 TUT06

Thm 7.2.1 (Cauchy Criterion)

A fcn $f:[a,b]\to\mathbb{R}$ belongs to $\mathcal{R}[a,b]$ iff
$\forall\, \varepsilon>0,\ \exists\, \eta_\varepsilon>0$ s.t.
if $\dot{P}$, $\dot{Q}$ are tagged partitions of $[a,b]$ with $\|\dot{P}\|, \|\dot{Q}\| < \eta_\varepsilon$,
then
$$\left|S(f;\dot{P}) - S(f;\dot{Q})\right| < \varepsilon$$

Thm 7.2.3 (Squeeze Thm)

Let $f:[a,b]\to\mathbb{R}$. Then $f\in\mathcal{R}[a,b]$ iff
$\forall\, \varepsilon>0,\ \exists$ fcns $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[a,b]$
with $\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \quad \forall\, x\in[a,b]$
s.t. $\displaystyle\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$

Thm 7.2.7 If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, then $f\in\mathcal{R}[a,b]$

Thm 7.2.8 (Additivity Thm)

Let $f:[a,b]\to\mathbb{R}$ and $c\in(a,b)$
Then $f\in\mathcal{R}[a,b] \iff f|_{[a,c]}\in\mathcal{R}[a,c]$ & $f|_{[c,b]}\in\mathcal{R}[c,b]$.
In this case,
$$\int_a^b f = \int_a^c f + \int_c^b f$$

4. If $\alpha(x) := -x$ and $\omega(x) := x$ and if $\alpha(x) \le f(x) \le \omega(x)$ for all $x \in [0, 1]$, does it follow from the Squeeze Theorem 7.2.3 that $f \in \mathcal{R}[0, 1]$?

Ans: $\int_0^1 (\omega - \alpha) = \int_0^1 2x = 1$, not small.

No. Counter-example:

Consider $f(x) := \begin{cases} x & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$

Then $\alpha(x) \le f(x) \le \omega(x) \ \forall x \in [0,1]$.

Recall: $h(x) := x \in \mathcal{R}[0,1]$ and $\int_0^1 h = \frac{1}{2}$

Let $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a uniform partition of $[0, 1]$

i.e. $x_i = i/n$, $i = 0, \dots, n$

Take tags $q_i = (x_{i-1} + x_i)/2 = \frac{2i-1}{2n} \in \mathbb{Q}$

Then the corresponding tagged partition $\dot{P}_1 = \{[x_{i-1}, x_i], q_i\}_{i=1}^n$ satisfies

$$S(f; \dot{P}_1) = \sum_{i=1}^n f(q_i)(x_i - x_{i-1}) = \sum_{i=1}^n \left(\frac{x_{i-1} + x_i}{2}\right)(x_i - x_{i-1})$$

$$= \frac{1}{2}\sum_{i=1}^n (x_i^2 - x_{i-1}^2) = \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}$$

OTOH, if we take tags $r_i \in [0,1] \setminus \mathbb{Q}$, then the corresponding tagged partition $\dot{P}_2 = \{[x_{i-1}, x_i], r_i\}_{i=1}$ satisfies

$$S(f; \dot{P}_2) = \sum_{i=1}^n f(r_i)(x_i - x_{i-1}) = 0$$

Since $\|\dot{P}_1\| = \|\dot{P}_2\| = 1/n$ can be arbitrarily small, while

$$|S(f; \dot{P}_1) - S(f; \dot{P}_2)| = \frac{1}{2},$$

it follows from Cauchy Criterion that $f \notin \mathcal{R}[0,1]$ //

11. If f is bounded by M on $[a, b]$ and if the restriction of f to every interval $[c, b]$ where $c \in (a, b)$ is Riemann integrable, show that $f \in \mathcal{R}[a, b]$ and that $\int_c^b f \to \int_a^b f$ as $c \to a+$. [*Hint:* Let $\alpha_c(x) := -M$ and $\omega_c(x) := M$ for $x \in [a, c)$ and $\alpha_c(x) := \omega_c(x) := f(x)$ for $x \in [c, b]$. Apply the Squeeze Theorem 7.2.3 for c sufficiently near a.]

Ans: $\forall\, c \in (a,b)$, define

$$\alpha_c(x) := \begin{cases} -M, & x \in [a,c) \\ f(x), & x \in [c,b] \end{cases}$$

$$\omega_c(x) := \begin{cases} M, & x \in [a,c) \\ f(x), & x \in [c,b] \end{cases}$$

Clearly, $\alpha_c(x) \le f(x) \le \omega_c(x) \quad \forall\, x \in [a,b]$

Need to check: $\alpha_c, \omega_c \in \mathcal{R}[a,b]$ and $\int(\omega_c - \alpha_c) < \varepsilon$

Note α_c is constant on $[a,c]$ except at $x = c$

$\Rightarrow \alpha_c \in \mathcal{R}[a,c]$ by Thm 7.1.3 $\left(\alpha_c\big|_{[a,c]} \in \mathcal{R}[a,c]\right)$

It is given that $\alpha_c \in \mathcal{R}[c,b]$.

By Additivity Thm 7.2.9, $\alpha_c \in \mathcal{R}[a,b]$.

Similarly, $\omega_c \in \mathcal{R}[a,b]$.

Now
$$\int_a^b (\omega_c - \alpha_c) = \int_a^c 2M + \int_c^b \left(f(x) - f(x)\right) = 2M(c-a)$$

So, given $\varepsilon > 0$, take $c \in (a,b)$ s.t. $c - a < \frac{\varepsilon}{2M}$, we get
$$\int_a^b (\omega_c - \alpha_c) < \varepsilon$$

By Squeeze Thm 7.2.3, $f \in \mathcal{R}[a,b]$

Moreover, for such c,
$$\left|\int_a^b f - \int_c^b f\right| = \left|\int_a^c f\right| \le M(c-a) = \frac{\varepsilon}{2} < \varepsilon.$$

Additivity Thm.

Hence $\int_c^b f \to \int_a^b f$ as $c \to a^+$.

11. If f is bounded by M on $[a, b]$ and if the restriction of f to every interval $[c, b]$ where $c \in (a, b)$ is Riemann integrable, show that $f \in \mathcal{R}[a, b]$ and that $\int_c^b f \to \int_a^b f$ as $c \to a+$. [*Hint:* Let $\alpha_c(x) := -M$ and $\omega_c(x) := M$ for $x \in [a, c)$ and $\alpha_c(x) := \omega_c(x) := f(x)$ for $x \in [c, b]$. Apply the Squeeze Theorem 7.2.3 for c sufficiently near a.]

12. Show that $g(x) := \sin(1/x)$ for $x \in (0, 1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0, 1]$.

Ans: Use Q11 !

Note: 1) $|g(x)| \le 1 \quad \forall x \in [0,1]$

So g is bounded on $[0,1]$

2) $\forall c \in (0,1)$

$\sin(1/x)$ is cts on $[c,1]$

$\Rightarrow g$ is cts on $[c,1]$

$\Rightarrow g \in \mathcal{R}[c,1]$ by Thm 7.2.7

Hence, by Q11, $g \in \mathcal{R}[0,1]$. //

18. Let f be continuous on $[a, b]$, let $f(x) \geq 0$ for $x \in [a, b]$, and let $M_n := \left(\int_a^b f^n\right)^{1/n}$. Show that $\lim(M_n) = \sup\{f(x) : x \in [a, b]\}$.

Ans: If $f \equiv 0$, the result is trivial.

Otherwise, since f is cts on $[a,b]$, EVT implies that $\exists\, x_0 \in [a,b]$ s.t. $f(x_0) = \sup\{f(x) : x \in [0,1]\} > 0$

Let $\varepsilon > 0$. s.t. $\varepsilon < f(x_0)$

By continuity, $\exists\, \delta > 0$ s.t

$$|f(x) - f(x_0)| < \varepsilon \quad \forall\, x \in (x_0-\delta, x_0+\delta) \cap [a,b].$$

In particular, $\exists\, [c,d] \subseteq [a,b]$ s.t

$$0 < f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon \quad \forall\, x \in [c,d].$$

Now,
$$\int_a^b f^n \leq \int_a^b f(x_0)^n = (b-a) f(x_0)^n$$

and
$$\int_a^b f^n \underset{f \geq 0}{\geq} \int_c^d f^n \geq \int_c^d \underbrace{(f(x_0) - \varepsilon)}_{>0}{}^n = (d-c)(f(x_0)-\varepsilon)^n$$

Hence, $(d-c)^{1/n}(f(x_0)-\varepsilon) \leq M_n = \left(\int_a^b f^n\right)^{1/n} \leq (b-a)^{1/n} f(x_0) \quad \forall\, n \in \mathbb{N}.$

Note $\lim_{n\to\infty} \alpha^{1/n} = 1 \quad \forall\, \alpha > 0.$

Letting $n \to \infty$,

$\exists\,?$

$$\cancel{f(x_0) - \varepsilon \leq \lim_{n\to\infty} M_n \leq f(x_0)}$$

$$f(x_0) - \varepsilon \leq \liminf_{n\to\infty} M_n \leq \limsup_{n\to\infty} M_n \leq f(x_0)$$

Letting $\varepsilon \to 0^+$, we have $\liminf_{n\to\infty} M_n = \limsup_{n\to\infty} M_n = f(x_0)$

That is $\lim_{n\to\infty} M_n = f(x_0) = \sup\{f(x) : x \in [0,1]\}$

//

19. Suppose that $a > 0$ and that $f \in \mathcal{R}[-a, a]$.
(a) If f is *even* (that is, if $f(-x) = f(x)$ for all $x \in [0, a]$), show that $\int_{-a}^{a} f = 2\int_0^a f$. No substitution yet!
(b) If f is *odd* (that is, if $f(-x) = -f(x)$ for all $x \in [0, a]$), show that $\int_{-a}^{a} f = 0$.

Ans: a) By Additivity Thm, $f \in \mathcal{R}[0,a]$, $f \in \mathcal{R}[-a,0]$.
$\forall \varepsilon > 0$, $\exists \eta > 0$ s.t. if $\dot{P}$, $\dot{Q}$ are tagged partition of $[0,a]$, $[-a,0]$ resp. with $\|\dot{P}\|, \|\dot{Q}\| < \eta$
then $|S(f;\dot{P}) - \int_0^a f| < \varepsilon$, $|S(f;\dot{Q}) - \int_{-a}^0 f| < \varepsilon$.

Let $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ be a tagged partition (say, uniform partition) of $[0,a]$ s.t. $\|\dot{P}\| < \eta$
Then $\dot{Q} := \{[-x_i, -x_{i-1}], -t_i\}_{i=1}^{n}$ is a tagged partition of $[-a,0]$ with $\|\dot{Q}\| = \|\dot{P}\| < \eta$
Moreover,

$$S(f|_{[0,a]}; \dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^{n} f(-t_i)[(-x_{i-1}) - (-x_i)]$$
$$= S(f|_{[-a,0]}; \dot{Q})$$

Now $\left|\int_0^a f - \int_{-a}^0 f\right| \leq \left|\int_0^a f - S(f;\dot{P})\right| + \cancel{\left|S(f;\dot{P}) - S(f;\dot{Q})\right|}^{\,0} + \left|S(f;\dot{Q}) - \int_{-a}^0 f\right|$
$< 2\varepsilon$

Since $\varepsilon > 0$ is arbitrary, we have $\int_0^a f = \int_{-a}^0 f$
Hence, by Additivity Thm, $\int_{-a}^{a} f = \int_{-a}^{0} f + \int_0^a f = 2\int_0^a f$

b) Follow the same argument but now using

$$S(f;\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^{n} -f(-t_i)[(-x_{i-1}) - (-x_i)] = -S(f;\dot{Q})$$

//

19. Suppose that $a > 0$ and that $f \in \mathcal{R}[-a, a]$.

(a) If f is *even* (that is, if $f(-x) = f(x)$ for all $x \in [0, a]$), show that $\int_{-a}^{a} f = 2\int_0^a f$.

(b) If f is *odd* (that is, if $f(-x) = -f(x)$ for all $x \in [0, a]$), show that $\int_{-a}^{a} f = 0$.

20. If f is continuous on $[-a, a]$, show that $\int_{-a}^{a} f(x^2)dx = 2\int_0^a f(x^2)dx$.

Ans: Note: 1) f cts on $[-a, a]$

$\Rightarrow g(x) := f(x^2)$ cts on $[-a, a]$

$\Rightarrow g \in \mathcal{R}[-a, a]$

2) $\forall x \in [0, a]$,

$g(-x) = f((-x)^2) = f(x^2) = g(x)$

So g is even

By Q19, $\int_{-a}^{a} g = 2\int_0^a g$

i.e. $\int_{-a}^{a} f(x^2)dx = 2\int_0^a f(x^2)dx$